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★**Heisenberg groups, theta functions and the Weil representation.**

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This text collects in 155 pages essential notions and facts from the Archimedean theory of the objects named in its title with some explicit proofs and several useful citations.

It is divided into two chapters. In the first one, for two positive integers m and n, the author introduces as his Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{(\lambda,\mu,\kappa)|\lambda,\mu\in\mathbb{R}^{(m,n)},\kappa\in\mathbb{R}^{(m,m)},\kappa+\mu^{\mathsf{T}}\lambda \text{ symmetric}\}$$

and describes its Lie algebra. For $\Omega\in\mathbb{H}_n$, the Siegel half-space, $W\in\mathbb{C}^{(m,n)}$, and S a positive definite symmetric real matrix of degree m, he defines the usual theta functions with characteristics $A,B\in\mathbb{R}^{(m,n)}$,

$$\vartheta^{(S)}\begin{bmatrix}A\\B\end{bmatrix}(\Omega,W)=\sum_{N\in\mathbb{Z}^{(n,m)}} e^{\pi i\sigma\{S((N+A)\Omega^t(N+A)+2(W+B)^t(N+A))\}}$$

and discusses these as functions on $\mathbb{C}^{(m,n)}$ with special transformation behaviour with respect to the lattice $L_\Omega=\mathbb{Z}^{(m,n)}\Omega+\mathbb{Z}^{(m,n)}$. The action of $H_{\mathbb{R}}^{(m,n)}$ on $\mathbb{C}^{(m,n)}$ is given along with the relevant associated automorphic factor. Sections 1.4 to 1.6 comprise the standard facts of the representation theory of $H_{\mathbb{R}}^{(n,m)}$ including the equivalence of the Schrödinger, the lattice and the Fock representations. Section 1.7 applies Kirillov's theory of coadjoint orbits to this (fairly simple) case. In section 1.8 the author refines the discussion of the Schrödinger representation to end up with the result that the Hermite polynomials form an orthonormal basis for the space $L^2(\mathbb{R}^{(m,n)},e^{-4\pi|\xi|^2}d\xi)$. Finally, in section 1.9, under the title "Harmonic analysis on the quotient space of $H_{\mathbb{R}}^{(n,m)}$", the author investigates the irreducible components of

$$L^2(H_{\mathbb{Z}}^{(n,m)}\diagdown H_{\mathbb{R}}^{(n,m)}).$$

He describes the connection among these irreducible components, explicitly describes the Schrödinger, the Fock, and the lattice representations, and provides orthonormal bases for each of these representation spaces.

The second chapter is devoted to theta functions and the Weil representation. It starts by reviewing the symplectic group $\mathrm{Sp}(n,\mathbb{R})$ and its action on the Siegel upper half-space $\mathbb{H}_n$. Using the Maslov index as in the book by Lion-Vergne, the author constructs the universal covering group of the symplectic group. In section 2.3 the Jacobi group is defined as the semi-direct product

$$G^J=\mathrm{Sp}(n,\mathbb{R})\ltimes H_{\mathbb{R}}^{(n,m)}.$$

For a symmetric positive definite real matrix c of degree m, using the Stone–von Neumann theorem and the introduction of the metaplectic cover $\mathrm{Mp}(n,\mathbb{R})_c$ of $\mathrm{Sp}(n,\mathbb{R})$, the author exhibits the formalism of the Weil representation ω_c in some detail, adapted to the situation here at hand. He reproduces the result of Kashiwara-Vergne decomposing ω_c into irreducibles as a representation of $\mathrm{Mp}(n,\mathbb{R})$. As they are needed here, pluriharmonic polynomials on $R^{(m,m)}$ come into play. In section 2.4 covariant maps for the Weil representation are constructed. Section 2.5 and section 2.6 treat theta series associated to positive definite quadratic forms and moreover those with plurihamonic polynomials as coefficients. Based on material from D. B. Mumford's books [*Tata lectures on theta. I*, Progr. Math., 28, Birkhäuser Boston, Boston, MA, 1983; MR0688651 (85h:14026); *Tata lectures on theta. II*, Progr. Math., 43, Birkhäuser Boston, Boston, MA, 1984; MR0742776 (86b:14017); *Tata lectures on theta. III*, Progr. Math., 97, Birkhäuser Boston, Boston, MA, 1991; MR1116553 (93d:14065)], it is proved that theta series with pluriharmonic polynomials as coefficients are modular forms for a suitable congruence subgroup of the Siegel modular group. In section 2.7 the relation between the Weil representation and theta series is investigated where the covariant maps from 2.4 become essential.

Finally, in section 2.8, called "Spectral theory on the abelian variety", for the principally polarized abelian variety A_Ω attached to $\Omega \in \mathbb{H}_n$, the author decomposes the L^2-space of A_Ω explicitly into irreducibles. Moreover, at the beginning of the section, the action of the Jacobi group G^J on the Siegel-Jacobi space

$$\mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)}$$

is investigated. A fundamental domain $\mathcal{F}_{n,m}$ for the action of

$$\Gamma_{n,m} := \Gamma_n \ltimes H_{\mathbb{Z}}^{(n,m)}$$

and explicit formulae for the invariant metric $ds^2_{n,m}$ and its Laplacian $\Delta_{n,m}$ are given. The author states that it might be interesting to investigate the spectral theory of $\Delta_{n,m}$ on $\mathcal{F}_{n,m}$, but he also remarks that this is very complicated and difficult at the moment; therefore, he only deals with the L^2-theory of the abelian variety mentioned above.

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